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# Rational collocation for linear boundary value problems

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*Abstract:* In this article some numerical methods of rational collocation for linear boundary value problems are studied. Several numerical examples are given.

*Keywords:* Rational collocation, boundary value problems, splines.

## 1. Introduction

Let $U$ and $V$ be functional spaces. We consider the equation

$$Tx = y, \tag{1.1}$$

where the domain $D(T)$ of $T$ is contained in $U$ and $y \in V$. Here we assume the existence of a unique solution to (1.1). A method of collocation consists of determining an element $x_n(t)$ of an $n$-dimensional subspace $U_n$ of $U$ such that the equality

$$Tx_n(t_i) = y(t_i) \tag{1.2}$$

holds at certain collocation knots $\{t_i\}_{i=1}^n$.

If the set $\{\phi_i\}_{i=1}^n$ is a basis of $U_n$, then

$$x_n(t) = \sum_{i=1}^{n} a_i \phi_i(t),$$

where the $a_i$'s are computed from equations (1.2).

If $T$ is a linear operator, then (1.2) is a linear algebraic system and the matrix $C = (c_{ij})$, with $c_{ij} = T\phi_j(t_i)$ is called *collocation matrix*.

A collocation method is said to be of the *polynomial type* if the $\phi_i$'s are either polynomials or polynomic splines, whereas it is said to be of the *rational type* if the $\phi_i$'s are rational functions or rational splines.

The most natural way to develop rational collocation methods seems to be a previous choice of the denominators of the $\phi_i$'s, as in the case of Padé-type approximants [2].

In this article, rational collocation is studied in connection with linear boundary value problems (BVP for short). Under the restriction that the denominator $q(s)$ of the $\phi_i$'s is the same polynomial (or piecewise polynomial) with no zeros on $[a, b]$, it is possible to link the BVP

$$Lu = f, \qquad F(u(a), u(b)) = 0 \tag{1.3}$$

with the associated BVP

$$L_q u = fq, \qquad F_q(u(a), u(b)) = 0, \tag{1.4}$$

such that finding an approximate solution of (1.3) with rational collocation is equivalent (in a sense which will be precised) to finding an approximate solution of (1.4) with polynomial collocation.

Moreover, the associated BVP permits to study the convergence of the rational method. The problem of an adequate choice of $q(s)$ will be considered in a work which is now being completed. For the moment, let us emphasize that our method seems quite promising in certain singular perturbation problems, as an alternative to the existing procedures [4,6]. Here we restrict ourselves to show several numerical examples in Section 3.

## 2. Rational collocation with a fixed denominator

We study the linear differential equation

$$L[u] = \sum_{k=0}^{m} e_k(s)u^{(k)}(s) = f(s), \quad a \leqslant s \leqslant b, \tag{2.1}$$

with the $m$ linearly independent boundary conditions

$$\sum_{k=0}^{m-1} \alpha_{ik}u^{(k)}(a) + \beta_{ik}u^{(k)}(b) = 0, \quad \alpha_{ik}, \beta_{ik} \in \mathbb{R}, \ 1 \leqslant i \leqslant m. \tag{2.2}$$

Let $\Pi_n$: $a = s_0 < s_1 < \cdots < s_n = b$ be a partition of $[a, b]$, and let $L(\Pi_n, k, m)$ be the set of piecewise polynomials of degree up to $k$ over each subinterval of $\Pi_n$.

Let $L_q(\Pi_n, m+d, m) = \{ p/q\colon\ p \in L(\Pi_n, m+d, m)\}$. We denote by $\overline{L}_q(\Pi_n, m+d, m)$ the set of functions in $L_q(\Pi_n, m+d, m)$ satisfying the boundary conditions (2.2). Let now $q(s)$ be a polynomial of fixed degree and consider the partition of [0, 1], $0 = t_0 < t_1 < \cdots < t_d = 1$. Now, define $nd+1$ points $s_{ij} = \xi_i(t_j)$, where $\xi_i(t) = s_i + t(s_{i+1} - s_i)$, $0 \leqslant i \leqslant n-1$. Under these conditions, we prove the following result, which essentially gives the equivalence of problems (1.3) and (1.4) in the general case, i.e., when $v$ is simply a sufficiently differentiable function in $[a, b]$.

**Theorem 1.** *Let $v \in C^{(m)}[a, b]$ and $L[u] = \sum_{k=0}^{m} e_k u^{(k)}$ be a linear differential operator of order $m$. Then*

$$L\left(\frac{u}{v}\right) = \frac{L_v u}{v}, \quad \text{where } L_v u = \sum_{k=0}^{m} f_k u^{(k)}. \tag{2.3}$$

*The $f_k$'s can be iteratively computed by*

$$f_{k-1} = -\sum_{j=1}^{h_k} \frac{v^{(j)}}{v} \frac{(k)_j}{j!} f_{k+j-1} + e_{k-1},$$

$$(k)_j = k(k+1)\cdots(k+j-1), \quad k=m,\ m-1,\ldots,1,\ h_k = m-k+1,$$

$$f_m = e_m.$$

**Proof.** By definition of $L_v$, the $f_k$'s satisfy

$$\frac{1}{v}\left(\sum_{j=0}^{h_k} v^{(j)} \frac{(k)_j}{j!} f_{k+j-1}\right) = e_{k-1};$$

note that

$$\frac{(k)_j}{j!} = \binom{k+j-1}{k-1}.$$

On the other hand,

$$\frac{1}{v} L_v u = \frac{1}{v} \sum_{k=0}^{m} f_k \left(\frac{u}{v} v\right)^{(k)} = \frac{1}{v}\left(f_0 u + \sum_{k=1}^{m} f_k \left(\sum_{r=0}^{k} \binom{k}{r} \left(\frac{u}{v}\right)^{(r)} (v)^{(k-r)}\right)\right) \tag{2.4}$$

is also true. Now, since

$$\frac{1}{v}\left(\sum_{j=1}^{h_k} v^{(j)} \frac{(k)_j}{j!} f_{k+j-1}\right)$$

are the coefficients of $(u/v)^{(k-1)}$ in (2.4), one can easily deduce that $(L_v u)/v = L(u/v)$. $\square$

Next, we give the explicit form of the coefficients in (2.3) for the important case where $v$ is a polynomial.

**Theorem 2.** *Under the same conditions as in Theorem* 1, *let $v \equiv q$ be a polynomial of degree $n$. Then,*

$$f_{k-1}(s) = -\frac{q'(s)}{q(s)} f_k(s) k - \frac{q''(s)}{q(s)} \frac{k(k+1)}{2!} f_{k+1}(s)$$

$$- \cdots - \frac{q^{(n)}}{q(s)} \frac{k(k+1)\cdots(n+k-1)}{k!} f_{k+n-1}(s) + e_{k-1}(s),$$

$$k = m,\ m-1,\ldots,1,$$

$$f_m(s) = e_m(s), \qquad f_{m+1}(s) = f_{m+2}(s) = \cdots = f_{m+n-1}(s) = 0.$$

**Proof.** The result follows from Theorem 1 by identifying $v$ and $q$, and from the fact that $q^{(r)} = 0$ for $r > n$. $\square$

The following result furnishes the relation between the systems arising from the collocation method for the BVPs involving operators $L$ and $L_q$.

**Theorem 3.** *Under the hypotheses of Theorem 2, there exist a finite number of constants* $\alpha'_{ik}$, $\beta'_{ik}$, $1 \leqslant i \leqslant m$, *such that the collocation matrix on* $L_q(\Pi_n, m+d, m)$ *for the problem* (2.1), (2.2) *is equivalent to the collocation matrix on* $L_q(\Pi_n, m+d, m)$ *for the problem*

$$L_q[v] = \sum_{k=0}^{m} f_k(s) v^{(k)}(s) = f(s) q(s), \tag{2.5}$$

$$\sum_{k=0}^{m-1} \alpha'_{ik} v^{(k)}(a) + \beta'_{ik} v^{(k)}(b) = 0, \quad 1 \leqslant i \leqslant m, \tag{2.6}$$

*when the same knots* $s_{ij}$ *are used in both BVPs.*

*The coefficients* $f_k$ *and the constants* $\alpha'_{ik}$, $\beta'_{ik}$ *depend on* $q(s)$ *and can be computed iteratively.*

**Proof.** Let $L_q$ be the linear operator constructed in Theorem 1, such that

$$L\left(\frac{u}{q}\right) = \frac{L_q u}{q}, \quad \text{for any } u \in C^{(m)}[a, b], \tag{2.7}$$

and also

$$\sum_{k=0}^{m-1} \alpha_{ik} \left(\frac{u}{q}\right)^{(k)}(a) + \beta_{ik} \left(\frac{u}{q}\right)^{(k)}(b) = \sum_{k=0}^{m-1} \alpha'_{ik} u^{(k)}(a) + \beta'_{ik} u^{(k)}(b), \tag{2.8}$$

where the coefficients $\alpha'_{ik}$, $\beta'_{ik}$ can be obtained iteratively from the coefficients $\alpha_{ik}$, $\beta_{ik}$ by invoking Theorem 1.

Hence, if $x = s_{ij}$ is a collocation knot, one has $L(u/q)(x) = f(x)$ if and only if $L_q u(x) = f(x) q(x)$. Also

$$\sum_{k=0}^{m-1} \alpha_{ik} \left(\frac{u}{q}\right)^{(k)}(a) + \beta_{ik} \left(\frac{u}{q}\right)^{(k)}(b) = 0 \quad \text{if and only if}$$

$$\sum_{k=0}^{m-1} \alpha'_{ik} u^{(k)}(a) + \beta'_{ik} u^{(k)}(b) = 0. \qquad \square$$

**Definition.** The problem (2.5), (2.6) will be called *associated boundary value problem* of (2.1), (2.2) with respect to $q$ (ABVP, for short).

The foregoing clearly shows that from the practical point of view, applying rational collocation to the BVP is equivalent to the use of polynomic collocation in the ABVP.

We now give several results connecting both problems. If one is interested in the use of a factorized polynomial $q$, the iteration of Theorem 1 gives the result as stated in the next theorem.

**Theorem 4.** *Let* $q_1$ *and* $q_2$ *be polynomials with no zeros in* $[a, b]$ *and such that* $q = q_1 q_2$. *Then, with the notation of Theorem* 1, $L_q = (L_{q_1})_{q_2} = (L_{q_2})_{q_1}$.

**Proof.** We have

$$L\left(\frac{u}{q}\right) = L\left(\frac{u}{q_1 q_2}\right) = \frac{1}{q_1 q_2} (L_{q_1})_{q_2}(u) = \frac{1}{q} (L_{q_1})_{q_2}(u) = \frac{1}{q} L_q u.$$

Hence, $(L_{q_1})_{q_2} = L_q$ and $(L_{q_2})_{q_1} = L_q$. $\square$

Next we compute the coefficients of (2.3) for the case where $q$ has a single real pole.

Table 1

| $q(s)$ | $s-\lambda$ | $(s-\alpha)^2+\beta^2$ | $(s-\lambda)(s-\mu)$ | $[s-\alpha)^2+\beta^2](s-\lambda)$ | $(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_h)$ $\cdot((s-\alpha_1)^2+\beta_1^2)\cdots((s-\alpha_r)^2+\beta_r^2)$ |
|---|---|---|---|---|---|
| $L_q$ | $L_\lambda$ | $L_{\alpha\beta}$ | $L_{\lambda,\mu}$ | $L_{\alpha\beta,\lambda}$ | $L_{\lambda_1,\lambda_2,\ldots,\lambda_h,\alpha_1\beta_1,\ldots,\alpha_r\beta_r}$ |

**Theorem 5.** *Let $\lambda$ be a real constant and $q(s)=s-\lambda$. If $L_\lambda$ is the operator associated to $L$ with respect to $q$, then*

$$L_\lambda[v] = \sum_{k=0}^{m} f_k v^{(k)},$$

*where the coefficients $f_k(s)$ satisfy the recurrence formula*

$$f_{k-1}(s) = -\frac{k}{(s-\lambda)} f_k(s) + e_{k-1}(s), \quad k=m,\ m-1,\ldots,1, \qquad f_m(s)=e_m(s).$$

**Proof.** It is a direct consequence of Theorem 2, when $q(s)=s-\lambda$. □

It is not difficult to see what happens when $q$ has two conjugated complex poles.

**Theorem 6.** *Let $\alpha$ and $\beta$ be real constants and $q(s)=(s-\alpha)^2+\beta^2$. If $L_{\alpha\beta}$ is the operator associated to $L$ with respect to $q$, then*

$$L_{\alpha\beta}[v] = \sum_{k=0}^{m} f_k v^{(k)},$$

*where the coefficients $f_k(s)$ satisfy the recurrence formula*

$$f_{k-1}(s) = -\frac{2(s-\alpha)}{(s-\alpha)^2+\beta^2} k f_k(s) - \frac{1}{(s-\alpha)^2+\beta^2} f_{k+1}(s) k(k+1) + e_{k-1}(s),$$

$$k=m,\ m-1,\ldots,1,$$

*where $f_m(s)=e_m(s)$, $f_{m+1}(s)=0$.*

**Proof.** It is enough to take $q(s)=(s-\alpha)^2+\beta^2$ in Theorem 2. □

It is now convenient to introduce some notation to characterize the associated operator $L_q$ for a given polynomial $q(s)$. In Table 1 we give the relevant cases.

A generalization of Theorems 5 and 6 is now given as a consequence of Theorem 4, for the case where $q$ is determined by its real and complex roots.

**Theorem 7.** *Let $q(s)$ be a polynomial with $h$ real roots $\lambda_1, \lambda_2, \ldots, \lambda_h$ and $2r$ complex conjugate roots $\alpha_k \pm \mathrm{i}\beta_k$, $k=1,2,\ldots,r$. Assume that the following equalities hold:*

$$f_k^{(0)}(s) = e_k(s), \quad k=0,1,\ldots,m,$$

$$f_{k-1}^{(j)}(s) = -\frac{2(s-\alpha_j)}{(s-\alpha_j)^2+\beta_j^2} k f_k^{(j)}(s) - \frac{1}{(s-\alpha_j)^2+\beta_j^2} f_{k+1}^{(j)}(s) k(k+1) + f_{k-1}^{(j-1)}(s),$$

$$k = m,\ m-1, \ldots, 1,$$

$$f_m^{(j)}(s) = e_m(s), \quad f_{m+1}^{(j)}(s) = 0, \qquad j = 1, 2, \ldots, r,$$

*and*

$$f_{k-1}^{(r+j)}(s) = \frac{-k}{s-\lambda_j} f_k^{(r+j)}(s) + f_{k-1}^{(r+j-1)}(s), \quad k = m,\ m-1, \ldots, 1,$$

$$f_m^{(r+j)}(s) = e_m(s), \quad j = 1, 2, \ldots, h.$$

*Then we have*

$$f_k(s) = f_k^{(r+h)}(s), \quad k = 0, 1, 2, \ldots, m.$$

**Proof.** By applying the operator $L$ to $p/q$, one has

$$L\left(\frac{p}{q}\right) = \sum_{k=0}^{m} f_k^{(0)} \left(\frac{p}{q}\right)^{(k)} = L_{\alpha_1\beta_1}\left(\frac{p\left[(s-\alpha_1)^2+\beta_1^2\right]}{q}\right) \frac{1}{(s-\alpha_1)^2+\beta_1^2}$$

$$= L_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3,\ldots,\alpha_r\beta_r}\left(\frac{p\left(\prod_{k=1}^{r}\left[(s-\alpha_k)^2+\beta_k^2\right]\right)}{q}\right) \frac{1}{\prod_{k=1}^{r}\left[(s-\alpha_k)^2+\beta_k^2\right]}$$

$$= L_{\alpha_1\beta_1,\ldots,\alpha_r\beta_r,\lambda_1}\left(\frac{p(s-\lambda_1)\left(\prod_{k=1}^{r}\left[(s-\alpha_k)^2+\beta_k^2\right]\right)}{q}\right)$$

$$\times \frac{1}{\prod_{k=1}^{r}\left[(s+\alpha_k)^2+\beta_k^2\right]} \frac{1}{(s-\lambda_1)}$$

$$= \frac{1}{q} L_{\alpha_1\beta_1,\ldots,\alpha_r\beta_r,\lambda_1,\ldots,\lambda_h}(p) = \frac{1}{q} L_q(p).$$

Note that the coefficients of the operator $L_{\alpha_1\beta_1,\ldots,\alpha_j\beta_j}$ are $f_k^{(j)}(s)$, $j = 1, 2, \ldots, r$, $k = 0, 1, \ldots, m$, and those of the operator $L_{\alpha_1\beta_1,\ldots,\alpha_r\beta_r,\lambda_1,\ldots,\lambda_j}$ are $f_k^{(r+j)}(s)$, $j = 1, 2, \ldots, h$, $k = 0, 1, \ldots, m$. □

## 3. Numerical examples

**Example 1.**

$$y'' = y'\left(1 - \frac{1}{x-1}\right) + \frac{y}{(x-1)^2},$$

$$y(-1) = -\frac{1}{2e}, \qquad y(0) = -1.$$

The exact solution of this problem is $y(x) = e^x/(x-1)$.

Table 2

| $n$ | $k$ | $x$ | Error |
|---|---|---|---|
| 2 | 2 | −0.23 | 0.00172294 |

Table 3
$n = 2$, $k = 2$

| $\alpha$ | $x$ | Error |
|---|---|---|
| 0.50 | −0.23 | 0.0008189 |
| 0.60 | −0.23 | 0.0006080 |
| 0.70 | −0.23 | 0.0004198 |
| 0.80 | −0.23 | 0.0002547 |
| 0.90 | −0.23 | 0.0001103 |
| 1.00 | −0.48 | −0.0000210 |
| 1.10 | −0.22 | −0.0001278 |
| 1.20 | −0.22 | −0.0002263 |
| 1.30 | −0.23 | −0.0003139 |
| 1.40 | −0.23 | −0.0003923 |
| 1.50 | −0.23 | −0.0004627 |
| 2.50 | −0.23 | −0.0009030 |
| 3.50 | −0.23 | −0.0011172 |

In the following tables, $n$ denotes the number of subintervals into which $[a, b]$ has been divided, $k$ stands for the number of collocation knots in each subinterval, obtained from the translation of the zeros of the $k$th Legendre polynomial (see [3] and [7, pp.304–316]).

For a numerical estimation of the error we have placed 101 points uniformly distributed over $[a, b]$, $x_i = a + \frac{1}{100}i(b-a)$, $0 \leqslant i \leqslant 100$. Let $x = x_i^*$ be the abscissa where the maximum error is reached.

The results shown in Table 2 correspond to the collocation method. The basis functions used for each $n$ and $k$ are cubic splines with knots at $\tau_j = a + jh$, $j = 0, 1, \ldots, nk-1$, where $h = (b-a)/(nk-1)$ [1,7].

The results shown in Table 3 correspond to the rational collocation method with a pole at the point $\alpha$. The numerator of the approximate solution is a cubic spline as in the previous case.

Table 4
$t = 1.1$, $p_a = -2$, $p_b = 1.101$, $n = 5$, $k = 2$

| $x$ | Approximate solution |
|---|---|
| −1.00 | 1.00000000 |
| −0.979 | 0.74539221 |
| −0.958 | 0.53971954 |
| −0.937 | 0.37700652 |
| 0.974 | 0.00602814 |
| 0.995 | 0.00756849 |
| 1.016 | 0.00988109 |
| 1.037 | 0.01372623 |
| 1.058 | 0.02134935 |
| 1.079 | 0.04356944 |
| 1.100 | 1.00000000 |

Table 5
$p_a = -1.001$, $p_b = 0.901$, $n = 5$, $k = 2$

| $x$ | Approximate solution |
|---|---|
| −1.00 | 1.00000000 |
| −0.981 | 0.02683993 |
| −0.962 | 0.00409645 |
| −0.943 | −0.00238314 |
| 0.824 | 0.01047539 |
| 0.843 | 0.01470289 |
| 0.862 | 0.02308673 |
| 0.881 | 0.04747287 |
| 0.900 | 1.00000000 |

Table 6
$t = 1.5$, $p_a = -2$, $p_b = 1.501$, $n = 5$, $k = 2$

| $x$ | Approximate solution |
|---|---|
| −1.00 | 1.00000000 |
| −0.975 | 0.74171862 |
| −0.950 | 0.53429704 |
| −0.925 | 0.37112942 |
| 1.375 | 0.00648292 |
| 1.400 | 0.00843389 |
| 1.425 | 0.01167924 |
| 1.450 | 0.01812254 |
| 1.475 | 0.03698876 |
| 1.500 | 1.00000000 |

**Example 2.** In [9] asymptotic expansions of the form

$$y(x,\, t,\, \epsilon) = \lambda(t,\, \epsilon)x + [\alpha(0) + \lambda(t,\, \epsilon)]\ \mathrm{e}^{-(x+1)/\epsilon} + [\beta(0) - \lambda(t,\, \epsilon)t]\ \mathrm{e}^{-t(t-x)/\epsilon} + \mathrm{O}(\epsilon)$$

are considered, where

$$\lambda(t,\, \epsilon) = \frac{\beta(0) - \alpha(0)\ \mathrm{e}^{-(t-1)/\epsilon}}{t + \mathrm{e}^{(t-1)/\epsilon}}.$$

in order to study the behaviour of the solution $y(x,\, t,\, \epsilon)$ near the boundary in certain differential problems, namely:

$$\epsilon y'' - xy' + y = 0,$$
$$y(-1) = \alpha(\epsilon), \qquad y(t) = \beta(\epsilon).$$

Now, making use of rational collocation with two poles $p_a$ and $p_b$ and ten knots of collocation obtained from the translation of the zeros of the Legendre polynomial of degree two, we have for $\alpha(\epsilon) = \beta(\epsilon) = 1$, $\epsilon = 10^{-3}$ and $t = 1.1$, the results of Table 4.

Here, a boundary layer in the right-hand side can be observed, which agrees with the qualitative description of [9].

Taking $t = 0.9$ we get the results of Table 5.

Note now a boundary layer in the left-hand-side of the interval, again in agreement with [9].

Finally, when $t = 1.5$, we have the results of Table 6.

Now a boundary layer in the right-hand-side is simulated quite precisely.

**Remark.** In the case of *nonlinear problems*, the choice of an ABVP is more difficult. We are presently considering some cases where analytical information of the solution is available. For example, boundary value problems like $-\epsilon y'' + a(y)y' = 0$, $y(0) = \gamma_0$, $y(1) = \gamma_1$, where $\gamma_0 < \gamma_1$ [5]. In many cases, it is possible to determine the (unique) limit function $Y$ when $\epsilon \to 0$, and this knowledge can be used in a rather obvious way in the choice of $q(s)$.

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